Special Section — Marine Controlled-Source Electromagnetic Methods

2D marine controlled-source electromagnetic modeling:
Part 1 — An adaptive finite-element algorithm

Yuguo Li¹ and Kerry Key¹

ABSTRACT

In Part 1 of this work, we develop an adaptive finite-element algorithm for forward modeling of the frequency-domain, marine controlled-source electromagnetic (CSEM) response of a 2D conductivity structure that is excited by a horizontal electric dipole source. After transforming the governing equations for the secondary electromagnetic fields into the wavenumber domain, the coupled system of two partial differential equations for the strike-parallel electric and magnetic fields is approximated using the finite-element method. The model domain is discretized using an unstructured triangular element grid that readily accommodates arbitrarily complex structures. A numerical solution of the system of linear equations is obtained using the quasi-minimal residual (QMR) method, which requires much less storage than full matrix inversion methods. We implement an automated adaptive grid refinement algorithm in which the finite-element solution is computed iteratively on successively refined grids. Grid refinement is guided by an a posteriori error estimator based on a recently developed gradient recovery operator. The error estimator uses the solution to a dual problem in order to bias refinement toward elements that affect the solution at the electromagnetic (EM) receiver locations and enables the computation of asymptotically exact solutions to the 2.5D partial differential equations. We validate the finite element formulation against the canonical 1D reservoir model and study the performance of the adaptive refinement algorithm. An example model study of a complex offshore structure of interest for petroleum exploration illustrates the utility of the adaptive finite-element method for CSEM modeling.

INTRODUCTION

The marine controlled-source electromagnetic (CSEM) method has long been used to study the electrical conductivity of the oceanic crust and upper mantle (Young and Cox, 1981; Evans et al., 1994; Constable and Cox, 1996; MacGregor et al., 1998; MacGregor et al., 2001). More recently, an intense commercial interest has arisen to apply the method to detect offshore hydrocarbon reservoirs (e.g., Eidesmo et al., 2002) and the method also has proven effective for characterization of gas hydrate–bearing shallow sediments (e.g., Weitemeyer et al., 2006). The CSEM technique employs a deep-towed horizontal electric dipole source to transmit low-frequency electromagnetic (EM) energy to an array of seabed receivers. The amplitude and phase of the electric and magnetic fields recorded by receivers will vary, depending on the electrical conductivity structure of the underlying seabed, the water depth, and geometric factors such as the source–receiver offset. With the rapid increase in the number of marine CSEM data sets collected worldwide, the need for improved interpretational tools has become evident.

One-dimensional modeling (Chave and Cox, 1982; Flosadottir and Constable, 1996) has been around the longest, but many offshore structures are more complex than can be accommodated by 1D methods. Three-dimensional methods using staggered-grid finite differences (e.g., Newman and Alumbaugh, 1995; Weiss and Constable, 2006) allow for multidimensional modeling of heterogeneous structures, yet these methods are bound by the limitations of the structured grid. In particular, only a limited amount of structural detail can be included before the computational demands become in-
tractable for a modern workstation. For EM induction in two-dimen-
sional conductivity structures, the problem is described as 2.5D be-
cause an EM source emits a 3D field. Several 2.5D EM solutions
have been presented (e.g., Stoye and Greenfield, 1976; Lee and
Morrison, 1985; Everett and Edwards, 1992). For the frequency-do-
main CSEM method using a horizontal electric dipole source, Un-
sworth et al. (1993) developed a 2.5D finite-element (FE) solution,
which was subsequently used for modeling a mid-ocean ridge
CSEM data set (MacGregor et al., 2001). More recently, isoparamet-
ric elements in a structured grid were used to handle topography for
the 2.5D problem (Mitsuhata, 2000).

In this paper, we revisit the 2.5D marine CSEM problem and
present a new FE code that utilizes several recently developed tools
for enhancing the performance of finite-element methods. Our code
implements a fully unstructured triangular element grid, which readily
allows models to include arbitrarily complex structures such as
seafloor bathymetry, dipping layers, and other features that are cum-
berson to approximate with structured grids. In particular, the un-
structured grid is very suitable for accommodating multiple scale
structures, such as many small heterogeneities imbedded in larger
regional features. For solving the linear system arising from the FE
equations, we utilize the matrix-free quasi-minimal residual (QMR)
approach (Weiss, 2001), which does not require explicit storage of
the coefficient matrix and, hence, requires less memory storage. We
also implement an automated adaptive grid refinement algorithm
(Key and Weiss, 2006) based on a recently developed a posteriori error
estimator (Ovall, 2004, 2006) that enables the computation of
asymptotically exact solutions to the 2.5D partial differential equa-
tions. This is achieved by iterative refinement of the FE grid from a
coarse starting grid to progressively more detailed grids until the de-
sired solution accuracy has been attained.

The paper is organized as follows. We derive the two coupled par-
tial differential equations for the strike-parallel components of elec-
tric and magnetic fields in the wavenumber domain. We then de-
scribe their numerical approximation using the finite-element method.
We present the adaptive grid refinement scheme and discuss how
the method is implemented numerically. Next, we validate the code
through detailed comparison of 1D layered model responses, and
study the performance of the adaptive refinement method. Finally,
we present an example CSEM model study of a complex offshore
structure of interest for petroleum exploration.

**GOVERNING EQUATIONS**

We consider a 2D marine conductivity model with strike direction
\( x \) and a horizontal electric dipole source located above the sea floor.
Assuming the time variation \( e^{-i\omega t} \), the governing equations for the
electric and magnetic fields (\( \mathbf{E} \) and \( \mathbf{H} \)) in the quasi-stationary
approximation are

\[
\nabla \times \mathbf{E} = i\omega \mu_0 \mathbf{H},
\]

\[
\nabla \times \mathbf{H} - \sigma \mathbf{E} = \mathbf{J}_s,
\]

where \( \mu_0 \) is the magnetic permeability of free space, \( \omega \) is the angular
frequency and \( \sigma \) is the electric conductivity, which is defined here to
vary only in the \((y,z)\) plane. \( \mathbf{J}_s \) is the source current distribution,
which is singular at the location of the dipole source and hence is dif-
ficult to simulate numerically. To eliminate the source term from the
FE approximation, the principle of superposition may be applied.
The electromagnetic fields can then be expressed in terms of the pri-
mary fields (\( \mathbf{E}' \) and \( \mathbf{H}' \)) induced by a horizontal electric dipole in a

**FE APPROXIMATION**

The method of weighted residuals (e.g., Zienkiewicz, 1977) is
used to derive the FE approximation. The homogeneous Dirichlet
boundary condition applies on the outer boundary of the model. On the interelement boundaries, the tangential components of both electric and magnetic fields, $E_i$ and $H_i$, must be continuous. From this, we can further derive the continuity of the tangential components of the transformed secondary fields $\tilde{E}_i$ and $\tilde{H}_i$ on such boundaries. $\tilde{E}_i$ and $\tilde{H}_i$ have the following form

\[
\tilde{E}_i = -\frac{i\omega\mu_0}{\gamma^2} \frac{\partial \tilde{H}_i}{\partial n} - \frac{ik_x}{\gamma^2} \frac{\partial \tilde{E}_i}{\partial y} n_x - \frac{\partial \tilde{E}_i}{\partial z} n_y + \frac{i\omega\mu_0\sigma_i}{\gamma^2} (\tilde{E}_i n_z - \tilde{E}_i n_y),
\]

\[
\tilde{H}_i = -\frac{ik_x}{\gamma^2} \left( \frac{\partial \tilde{H}_i}{\partial y} n_x - \frac{\partial \tilde{H}_i}{\partial z} n_y \right) - \frac{\sigma}{\gamma^2} \frac{\partial \tilde{E}_i}{\partial n} - \frac{ik_x\sigma_i}{\gamma^2} \tilde{E}_i \cdot n,
\]

where $n$ denotes the outward normal unit vector of the boundary interface, $n_x$ and $n_y$ are the components of $n$ along the $y$- and $z$-axes, respectively.

Equation 11 is multiplied by an arbitrary variation of the transformed electric field $\delta\tilde{E}_i$ and integrated over the model area $\Omega$

\[
\int_{\Omega} \left[ \nabla \cdot \left( \frac{\sigma}{\gamma^2} \nabla \tilde{E}_i - \sigma \tilde{E}_i \right) \right] \delta\tilde{E}_i d\Omega
+ \int_{\Omega} \left[ -\frac{\partial}{\partial y} \left( \frac{ik_x}{\gamma^2} \frac{\partial \tilde{H}_i}{\partial y} \right) + \frac{\partial}{\partial z} \left( \frac{ik_x}{\gamma^2} \frac{\partial \tilde{H}_i}{\partial z} \right) \right] \delta\tilde{E}_i d\Omega
= \int_{\Omega} \left[ \frac{\partial}{\partial y} \left( \frac{ik_x}{\gamma^2} \tilde{E}_i \right) - \frac{\partial}{\partial z} \left( \frac{ik_x}{\gamma^2} \tilde{E}_i \right) \right] \delta\tilde{E}_i d\Omega
+ \sigma \tilde{E}_i \delta\tilde{E}_i d\Omega.
\]

Using the vector formula

\[
a \nabla \cdot b = \nabla \cdot \left( ab \right) - \nabla \cdot a \cdot b,
\]

the divergence theorem

\[
\int_{\Omega} \nabla \cdot b d\Omega = \int_{\Gamma} b \cdot n d\Gamma,
\]

and the Green’s theorem (Zienkiewicz, 1977)

\[
\int_{\Omega} \frac{\partial \phi}{\partial y} d\Omega = -\int_{\Omega} \phi \frac{\partial \phi}{\partial y} d\Omega + \int_{\Gamma} \phi \phi n_y d\Gamma,
\]

\[
\int_{\Omega} \frac{\partial \phi}{\partial z} d\Omega = -\int_{\Omega} \phi \frac{\partial \phi}{\partial z} d\Omega + \int_{\Gamma} \phi \phi n_z d\Gamma,
\]

equation 15 can be written in the equivalent form

\[
\int_{\Omega} \frac{\sigma}{\gamma^2} \nabla \tilde{E}_i \cdot \nabla \delta\tilde{E}_i d\Omega + \int_{\Omega} \sigma \tilde{E}_i \delta\tilde{E}_i d\Omega
- \int_{\Omega} \left[ \frac{\partial}{\partial y} \left( \frac{ik_x}{\gamma^2} \frac{\partial \tilde{H}_i}{\partial y} \right) + \frac{\partial}{\partial z} \left( \frac{ik_x}{\gamma^2} \frac{\partial \tilde{H}_i}{\partial z} \right) \right] d\Omega
= -\int_{\Omega} \sigma \tilde{E}_i \delta\tilde{E}_i d\Omega - \int_{\Gamma} \delta\tilde{E}_i \tilde{H}_i d\Gamma
- \int_{\Omega} \frac{ik_x}{\gamma^2} \nabla \delta\tilde{E}_i \cdot \nabla d\Omega.
\]

In formulating equation 20, we have used equation 14 to convert the integrand of the line integrals into $\delta\tilde{E}_i;\tilde{H}_i$.

Similarly, equation 12 is multiplied by an arbitrary variation of the magnetic field $\delta\tilde{H}_i$ and integrated over the region $\Omega$, then modified using the divergence theorem and the Green’s theorem, leading to the integral equation

\[
-\int_{\Omega} \frac{i\omega\mu_0}{\gamma^2} \nabla \tilde{H}_i \cdot \nabla \delta\tilde{H}_i d\Omega - \int_{\Omega} i\omega\mu_0\tilde{H}_i \delta\tilde{H}_i d\Omega
+ \int_{\Omega} \left\{ \frac{\partial}{\partial y} \left( \frac{ik_x}{\gamma^2} \frac{\partial \tilde{E}_i}{\partial z} \right) - \frac{\partial}{\partial z} \left( \frac{ik_x}{\gamma^2} \frac{\partial \tilde{E}_i}{\partial y} \right) \right\} d\Omega
= \int_{\Omega} \frac{i\omega\mu_0\sigma_i}{\gamma^2} \left( \tilde{E}_i \frac{\partial \delta\tilde{H}_i}{\partial y} - \tilde{E}_i \frac{\partial \delta\tilde{H}_i}{\partial z} \right) d\Omega + \int_{\Gamma} \delta\tilde{E}_i \tilde{H}_i d\Gamma.
\]

The model area $\Omega$ is subdivided into $n_e$ triangular finite elements with $n_e$ nodal vertices. The integrals of equations 20 and 21 thus decompose into integrals for each element, numbered with index $e = 1,2,\ldots, n_e$,

\[
\sum_{e=1}^{n_e} \sigma \int_{\Gamma_e} \nabla \cdot \delta\tilde{E}_e d\Gamma + \sum_{e=1}^{n_e} \sigma \int_{\Gamma_e} \delta\tilde{E}_e \cdot \nabla d\Gamma + \sum_{e=1}^{n_e} \sigma \int_{\Gamma_e} \delta\tilde{E}_e \cdot \nabla d\Gamma
- \sum_{e=1}^{n_e} \int_{\Gamma_e} \phi \delta\tilde{H}_e d\Gamma
\]

and

\[
-\sum_{e=1}^{n_e} \int_{\Gamma_e} \frac{i\omega\mu_0}{\gamma^2} \nabla \tilde{H}_e \cdot \nabla \delta\tilde{H}_e d\Gamma - \sum_{e=1}^{n_e} \int_{\Gamma_e} i\omega\mu_0 \tilde{H}_e \delta\tilde{H}_e d\Gamma
+ \sum_{e=1}^{n_e} \int_{\Gamma_e} \left\{ \frac{\partial}{\partial y} \left( \frac{ik_x}{\gamma^2} \frac{\partial \tilde{E}_e}{\partial z} \right) - \frac{\partial}{\partial z} \left( \frac{ik_x}{\gamma^2} \frac{\partial \tilde{E}_e}{\partial y} \right) \right\} d\Gamma
= \sum_{e=1}^{n_e} \int_{\Gamma_e} \frac{i\omega\mu_0\sigma_i}{\gamma^2} \left( \tilde{E}_e \frac{\partial \delta\tilde{H}_e}{\partial y} - \tilde{E}_e \frac{\partial \delta\tilde{H}_e}{\partial z} \right) d\Gamma + \sum_{e=1}^{n_e} \int_{\Gamma_e} \delta\tilde{E}_e \tilde{H}_e d\Gamma.
\]

The homogeneous Dirichlet boundary conditions are applied.

Within each triangular element, the transformed secondary fields ($\tilde{E}_i$ and $\tilde{H}_i$) are approximated as linear functions of $y$ and $z$ by interpolation of the three nodal values. The area integrals over each element in equations 22 and 23 are then evaluated analytically. Summing up the integrals over all the elements and assembling the $6 \times 6$ element matrices (resulting from the two coupled $3 \times 3$ systems) into a global system matrix, we obtain a linear equation system:
\[ \mathbf{K} \mathbf{u} = \mathbf{p}, \]  

(24)  

where \( \mathbf{u} \) is the column vector of order of \( 2n_e \) (\( n_e \) is the total number of vertices in \( \Omega \)), consisting of the unknown transformed secondary fields \( (\mathbf{E}_s^e, \mathbf{H}_s^e) \) at all vertices, and \( \mathbf{p} \) is the known vector resulting from the terms of the right side of equations 22 and 23. Once the strike-parallel components of secondary fields have been found in the wavenumber domain \( (k_x, y, z) \), they can be then transformed into the space domain \( (x, y, z) \) by inverse Fourier transformation and added to the primary fields to obtain the electromagnetic fields.

**ADAPTIVE GRID REFINEMENT USING A POSTERIORI ERROR ESTIMATION**

The error in the FE solution scales with the element size, and reciprocally, the solution accuracy increases with grid refinement. One method of ensuring an accurate solution is to create an FE grid with sufficiently small elements everywhere. However, this approach often results in a large linear system that is severely inefficient and might be well beyond the memory capacity of modern workstations. What is needed is a robust, numerically inexpensive approach often results in a large linear system that is severely inefficient yet capable of an accurate FE solution. A standard approach utilized in the engineering community is adaptive grid refinement using an a posteriori error estimator (e.g., Babuska et al., 1997; Oden et al., 1998). This method was recently shown to be effective for the 2D MT problem (Key and Weiss, 2006). Below we briefly review this approach and show how it can be applied to the 2.5D CSEM problem.

A recently developed class of a posteriori methods is called recovery-based error estimation. Recovery methods build upon the FE solution gradient \( \nabla \mathbf{u}_h \) being less accurate than the solution \( \mathbf{u}_h \). Although \( \mathbf{u}_h \) is piecewise linear in \( \Omega \), \( \nabla \mathbf{u}_h \) is only piecewise constant. A variety of averaging or projection techniques can be used to form an improved or recovered gradient, denoted \( \mathbf{R} \nabla \mathbf{u}_h \). The \( L_2 \)-norm of the difference between the recovered gradient and the piecewise constant gradient describes a local error indicator for a given element \( e \):

\[ \eta_e = \| (\mathbf{R} - I) \nabla \mathbf{u}_h \|_{L_2(e)}, \]  

(25)  

where \( I \) is the identity operator. A number of gradient recovery techniques are presented (e.g., Ainsworth and Oden, 2000). For general unstructured grids, Bank and Xu (2003) present a superconvergent gradient recovery operator \( \mathbf{R} = S^m Q_{\text{sc}} \), where \( Q_{\text{sc}} \) is the \( L_2 \) projection operator, \( \delta \) is an appropriate smoothing operator, and \( m \) is some appropriate number of iterations of the smoothing process, which is chosen to be two for our model tests. Bank and Xu (2003) demonstrated that the posteriori error estimator,

\[ \eta_e = \| (S^m Q_{\text{sc}} - I) \nabla \mathbf{u}_h \|_{L_2(e)}, \]  

(26)  

is asymptotically exact for the true gradient error as the element size decreases.

Grid refinement based on the local error indicator in equation 26 is suitable for generating a globally accurate solution, hence we will refer to this method as global refinement. However, in many situations the solution accuracy is required only at a few selected locations (i.e., the EM receivers) within the domain \( \Omega \), and several refinement iterations may have little impact on the solution accuracy at these possibly distant locations, as shown in Key and Weiss (2006). Manually increasing the refinement in the proximity of the receivers (i.e., local refinement) would not necessarily work because it is well known that the solution for the elliptic problem depends on data throughout the entire domain (e.g., Babuska et al., 1997; Oden et al., 1998). If we wish to reduce the local error in our approximation, grid refinement can not be restricted only to the local region. Global influences on the local error should somehow be taken into consideration (Ovall, 2006). An efficient solution to this dilemma is to apply a weighting term to the error indicator, where the weight is determined by a dual or adjoint solution of the FE system, referred to as dual error estimate weighting (DEW, e.g., Ovall, 2004, 2006; Key and Weiss, 2006) or goal-oriented error estimation (Rannacher and Stuttmeier, 1998; Prudhomme and Oden, 1999).

Consider a functional \( G \) that is some measure of the solution error \( u - u_h \), where \( u \) is the true solution of the partial differential equation and \( u_h \) is the finite element approximation. Since \( u \) is generally unknown, the DEW method uses a dual problem to approximate \( G \). Using inner product notation, we denote the operations on the left side of equations 20 and 21 as the primal operator \( B \), then equations 20 and 21 can be expressed as

\[ B(u, v) = F(v), \]  

(27)  

where

\[ B(u, v) = \int_{\Omega} \left[ \alpha \nabla u \cdot \nabla v + \eta uv \right] d\Omega - \int_{\Omega} \left[ \nabla_x (\beta \nabla_y p) - \nabla_y (\beta \nabla_x p) \right] d\Omega \]  

(28)  

with

\[ u = \mathbf{\hat{E}}_s^e, \quad v = \delta \mathbf{\hat{E}}_s^e, \quad p = \mathbf{\hat{H}}_s^e, \quad \alpha = \frac{\sigma}{\gamma^2}, \quad \eta = \sigma, \]  

for equation 20;  

\[ u = \mathbf{\hat{H}}_s^e, \quad v = \delta \mathbf{\hat{H}}_s^e, \quad p = \mathbf{\hat{E}}_s^e, \quad \alpha = \frac{i \omega \mu_0}{\gamma^2}, \quad \eta = i \omega \mu_0 \]  

(29)  

for equation 21;

\[ \beta = \frac{ik_0}{\gamma^2}, \quad \nabla_y = \frac{\partial}{\partial y}, \quad \nabla_z = \frac{\partial}{\partial z}. \]  

(30)  

To characterize the functional \( G \), we solve the dual problem

\[ B'(w, v) = G(v), \]  

(32)  

for \( w \), in which \( B' \) is a dual or adjoint operator and is defined as \( B'(w, v) = B(v, w) \). We then have

\[ G(u - u_h) = B'(w, u - u_h) = B(w - u_h, w - u_h), \]  

(33)  

where \( w \) and \( w_h \) are the true solution and the FE solution of the dual problem, respectively. In deriving the last term on the right side of equation 33, we have used the orthogonality property \( B(u - u_h, w_h) = F(w_h) - B(u_h, w_h) = 0 \). Provided with FE solutions \( u_h \) and \( w_h \), the
right side of equation 33 can be used to compute the equivalent solution to the error functional $G$:

$$B(u - u_h, w - w_h) = \int_{\Omega} \left[ \alpha \nabla (u - u_h) \cdot \nabla (w - w_h) 
+ \eta (u - u_h)(w - w_h) \right] d\Omega 
+ \int_{\Omega} \beta \nabla v_h \cdot \nabla p \n- \nabla_s (w - w_h)(\nabla_s p) \right] d\Omega. \quad (34)$$

Approximating the gradient terms in equation 34 by

$$\nabla (u - u_h) \approx (\mathbf{R} - I) \nabla u_h, \quad \nabla (w - w_h) \approx (\mathbf{R} - I) \nabla w_h, \quad (35)$$

and noting that $\alpha > \eta$ for expected conductivities and the low frequencies of marine CSEM, yields

$$B(u - u_h, w - w_h) = \int_{\Omega} \alpha (\mathbf{R} - I) \nabla u_h \cdot (\mathbf{R} - I) \nabla w_h d\Omega 
- \int_{\Omega} \beta (\mathbf{R} - I) \nabla w_h \cdot \nabla p \n- (\mathbf{R} - I) \nabla_s w_h \cdot \nabla_s p \right] d\Omega. \quad (36)$$

Equation 36 is an approximation of the error functional $G$, which can be computed using $u_h$ and $w_h$, the solutions of the primal and dual problem, respectively. From equation 36, we define the coupled DEW error indicator as

$$\hat{\eta}_e = \eta_e \tilde{\eta}_e + \nu_E^F + \nu_H^F, \quad (37)$$

where

$$\tilde{\eta}_e = \alpha \| (\mathbf{R} - I) \nabla w_h \|_{L_2(e)}, \quad (38)$$

and $\nu_E^F$ contains the coupling terms

$$\nu_E^F = \beta \| (\mathbf{R} - I) \nabla w_h \|_{L_2(e)} 
+ \beta \| (\mathbf{R} - I) \nabla w_h \|_{L_2(e)}. \quad (39)$$

The superscripts $E$ and $H$ refer to the electric and magnetic portions of the vectors. The superscript terms for $E$ and $H$ are reversed for $\nu_H^F$.

We note that this formulation differs from previous works (e.g., Ovall, 2004, 2006; Key and Weiss, 2006) in that the CSEM formulation includes the coupling terms.

To compute $w_h$, we need to solve the dual problem with a sensible choice for the error functional $G$. We use an approximation of the $H_1$ seminorm of $u_h$ (Ovall, 2004), as also used for the 2D MT problem. The dual problem is then written as

$$B'(v, w) = G(v) = \int_{\Omega} (\mathbf{R} - I) \nabla u_h \cdot \nabla v d\Omega, \quad (40)$$

where $\Omega_s$ is the possibly discontinuous subdomain of elements that contain EM receivers and $v$ is an arbitrary vector spanning the space of the finite elements. Outside $\Omega_s$, the value of $G$ is defined to be zero. We note that other choices for $G$ are possible, but that a study of optimal error functionals is beyond the scope of this work.

To summarize, the adaptive refinement proceeds as follows. First, a coarse grid is generated and the FE solution is calculated. The local error indicator $\eta_e$ (equation 25) is calculated for each element and the corresponding FE dual problem (equation 40) is solved for $w_h$. The DEW error indicator (equation 37) is computed and finally, the grid is refined where elements have large values of $\hat{\eta}$. An FE solution for the refined grid is computed and the process repeats iteratively until the solution converges.

**NUMERICAL IMPLEMENTATION**

We have implemented this formulation in a FORTRAN-90 code named MARE2DCSEM (Modeling with Adaptively Refined Elements) and describe some relevant details below. The Fourier-transformed primary fields used in the right side of equations 22 and 23 are calculated in a similar manner to Unsworth et al. (1993), except that we include an air layer in our formulation. For the Fourier transformations involved in the primary field calculation, we use the digital filter convolution code DLAGF0 (Anderson, 1982), since it is both fast and accurate. However, adaptive quadrature methods (e.g., Chave, 1983) could also be used.

The linear system in equation 24 is solved using the coupled two-term recurrence quasi-minimal residual (QMR) method (Freund and Nachtigal, 1994), using either a matrix-free formulation by Weiss (2001) or a modified version that precomputes and stores the matrix $K$ in a sparse form. The matrix-free QMR method requires only the products of the system matrix $K$ with a vector but not explicit storage of the matrix itself, and hence, it requires much less computer memory than full matrix inversion methods and about half the memory of our stored matrix QMR method. The QMR solver approach, along with increasing computer memory availability, led us to formulate equations 22 and 23 as a single coupled system (equation 24) as previously formulated for time-domain CSEM (Everett and Edwards, 1992), whereas computer limitations of the previous decade led Unsworth et al. (1993) to solve for $E$ and $H$ separately in an iterative manner.

Since the wavenumber spectra vary smoothly, we reduce the computational time required for the refinement iterations by computing the FE solution and error indicator for only a few logarithmically spaced wavenumbers. The error indicators for each wavenumber are then summed to give the total error indicators for the current grid. The gradient of the magnetic field can be discontinuous at conductivity boundaries and so the projection and smoothing operations are computed over subdomains of constant conductivity. The $L_2$ projection operator $O$ involves the solution of a coefficientless mass-matrix system which is easily solved using QMR. For the smoothing operator $S^n$, we apply $m$ iterations of QMR to the system $\nabla^2 (Q_e \nabla u_h) = 0$. Bank and Xu (2003) recommend $m = 2$ and we find that this works adequately for the CSEM problem.

Grid refinement is done in a similar manner as described for the 2D MT problem (Key and Weiss, 2006) by using external calls to Triangle (Shewchuk, 1996, 2002), an open-source constrained and conforming Delaunay triangulation code. We constrain the triangulation to have triangles with inner angles greater than 20°. Angles smaller than this can seriously degrade the performance of the QMR solver. Elements flagged for refinement are set to have their areas reduced by a factor of two, although the algorithm used by Triangle refines based on a hierarchy of vertices instead of triangles, so this is
only approximate. Solution convergence is measured by computing the relative difference between the current and previous grid’s \( \hat{E}_i \) and \( \hat{H}_i \) solutions at the receiver locations. For example, the relative differences \( \delta \hat{E}_{i,j,k} \) in the scattered electric field for refinement grid number \( i \), site number \( j \) and wavenumber \( k \), is computed using

\[
\delta \hat{E}_{i,j,k} = \frac{|\hat{E}_{i,j,k} - \hat{E}_{i-1,j,k}|}{|\hat{E}_{i,j,k}|},
\]

and similarly for \( \delta \hat{H}_{i,j,k} \). Refinement iterations cease when the maximum of \( \delta \hat{E} \) and \( \delta \hat{H} \) for all wavenumbers and sites falls below a user-defined threshold. The final grid is then used to compute the solution over a denser spectrum of wavenumbers. We have found that about 5–10 wavenumbers per \( \log_{10} \) decade, spaced evenly from about \( 10^{-5} \) to \( 10^{-1} \) m\(^{-1} \), is sufficient for computations involving the fields in the plane of the transmitter at \( x = 0 \). Further recommendations for the density and bandwidth of wavenumbers are given in Unsworth et al. (1993).

Once the strike-parallel components of the secondary fields have been found in the wavenumber domain \((k_x, y, z)\) for the final grid, they are added to the primary fields and then transformed into the spatial domain \((x, y, z)\) by inverse Fourier transformation using the digital filter convolution code DLAGF0 (Anderson, 1982). The integrand of the cosine transform is evaluated by cubic-spline interpolation of the fields computed at the discrete wavenumbers. The remaining \( y \) and \( z \) components can be computed using equations 5–10. For the terms involving spatial gradients, we use the recovered gradients \( \nabla \psi \) because they are more accurate than the piecewise constant FE gradients.

VALIDATION AND PERFORMANCE OF ADAPTIVE REFINEMENT

We validate the finite-element code and study the performance of the adaptive refinement algorithm by comparison of broadside and inline transmissions for a 1D model. The accuracy of 1D codes (e.g., Flosadottir and Constable, 1996) is probably unsurpassed by existing 3D codes, making 1D a useful tool for testing the capability of the 2D code. This is particularly so when considering all six EM field components at closely spaced intervals and at large distances from the transmitter. This is primarily because the 1D code uses an explicit formulation for each field component at each source-receiver offset, and thus attains a similar order of accuracy for each component. For verifying the finite-element code with a 1D model, we use a different background model for the primary field calculation such that large secondary fields are generated. Here, we only consider the fields in the plane of the transmitter at \( x = 0 \). However, we note that at along strike distances greater than a few kilometers, a much greater density of wavenumbers is required and the solution accuracy can be complicated by singularities in \( \hat{E}_i \) and \( \hat{H}_i \).

The test model is the canonical 1D reservoir (Constable and Weiss, 2006), which consists of 1 km of 0.3 \( \Omega \)m seawater, a 1-\( \Omega \)m sedimented seafloor, and a 100-m thick, 100-\( \Omega \)m reservoir layer at 1-km depth. For the finite-element modeling, the primary field model includes only the air, seawater, and background sediment layers, and so large secondary fields are generated by the resistive layer at 1-km depth. Both broadside (transmitter pointing along \( x \)) and inline (transmitter pointing along \( y \)) transmissions were tested at a frequency of 0.25 Hz, position \( x, y = 0 \) km, and height of 100 m above the seafloor. A linear array of 100 receivers was positioned along the seafloor at \( y \) ranges of 0–20 km. Both transmission geometries used the same coarse starting grid of 60-km width and 40-km height, containing 1909 vertices and 3766 elements. Adaptive refinement iterations were set to refine 5% of elements with largest DEW error indicator computed at wavenumbers \( k = 0.00001, 0.0001 \), and 0.001 m\(^{-1} \). Refinement iterations were stopped when \( \hat{E}_i \) and \( \hat{H}_i \) at the receivers varied by no more than 1% between subsequent iterations. The final refined grid was used to compute the spatial domain fields using 41 logarithmically spaced wavenumbers from \( 10^{-6} \) to \( 10^{-1} \) m\(^{-1} \).

The computed EM fields from both the 1D and finite-element codes are shown in Figure 1. The broadside solution converged to 1% after 18 grid refinements, resulting in a grid of 17,437 vertices and 34,815 elements. The inline solution converged after 20 refinements and required a grid of 23,237 vertices and 46,418 elements. Figure 2 shows the relative error of the finite element fields. \( E_i \) is generally accurate to better than about 0.1% in both amplitude and phase, and \( H_i \) is accurate to better than 1%. Since the \( y \) and \( z \) components are computed from less accurate spatial gradients of the \( x \) components, it is expected that their accuracy should be lower and this is generally observed for the broadside transmission, particularly at distances of 5–12 km where the \( y \) and \( z \) fields are about 10 times less accurate than the \( x \) field. This is probably due in part to the rapidly varying phase of the fields at these ranges, as shown in Figure 1. However, for the inline transmission the \( y \) and \( z \) fields have the same order of accuracy as the \( x \) field. In general, all components are within an order of magnitude accuracy, with the \( z \) component being the least accurate. The final grid sizes for both transmission modes present only a modest computational burden for a modern workstation and additional grid refinement iterations could be performed to further decrease the solution error.

Another validation test was to compute the full spatial domain solution at each grid refinement iteration and to compare this to the 1D code solution in order to assess the accuracy of our code from coarse to fine model discretization. The results of this test for an inline transmission of 0.25 Hz over the canonical 1D reservoir model are shown in Figure 3 (similar results are obtained for the broadside transmission). We conducted this test using two different styles of grid refinement: DEW-based refinement and a local refinement algorithm that approximately subdivides the elements containing receivers in half upon each grid refinement. We computed both refinement methods twice: once using a background primary field model with seafloor resistivity of 1 \( \Omega \)m (the same as the 1D model seafloor) and another time using a 0.3-\( \Omega \)m seafloor resistivity. For a 1-\( \Omega \)m primary model seafloor, the only elements containing a nonzero \( \sigma \) are contained within the resistive layer at 1 km beneath the seafloor. By changing the primary model seafloor to 0.3 \( \Omega \)m, all elements beneath the seafloor now have nonzero \( \sigma \), and correspondingly the FE solution requires more elements to attain a given level of accuracy. Figure 3 shows the mean relative error for each refinement method and primary model seafloor resistivity. Local refinement is effective for reducing the error for the first several refinements. However, the effect of pollution by errors in nonlocal elements is illustrated where the local refinement relative error asymptotes to a finite nonzero value for larger grids. As expected, local refinement with a 0.3-\( \Omega \)m primary model seafloor is less effective than for 1 \( \Omega \)m. In contrast, both DEW refinements attain much lower errors and continue to reduce the error as the number of grid vertices increases, showing that pollution from nonlocal elements is important and suggesting that DEW refinement can be used to compute asymptotically exact FE solutions. In this example, it might be concluded that local refinement
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giving a mean relative error of only about 1% and a maximum error of about 10% (not shown) is good enough for practical purposes. However, the simplicity of the canonical model suggests that local refinement will be much less effective for much more heterogeneous models, which may contain significant pollution errors from nonlocal elements.

Figure 4 shows the mean error when using the grid optimized for 0.25 Hz to compute solutions at other frequencies. For both broadside and inline transmissions, the grid produces solutions of similar accuracy at lower frequencies, but is less accurate at higher frequencies. This suggests that grids refined to about 0.1% error can be used to compute solutions of less than about 1% error for frequencies up to about an order of magnitude higher, and can be used for any lower frequency, although more complicated models may further restrict these limits. The most time-consuming part of our code is the solution of equation 24. Figure 5 shows a comparison of the time spent for solving the linear system using both the stored and matrix-free versions of the QMR solver on an Apple 2.2 GHz G5 computer. The times shown are the total times spent for each grid using all 41 wavenumbers. The matrix-free QMR method requires about 20% more time than the stored matrix QMR. For the stored matrix QMR, about 30% of the total time is used to compute and sparsely store the global system matrix $K$ in equation 24. Both stored and matrix-free QMR methods have a quadratic increase in time duration with the number of grid vertices.

The above examples have shown the performance of the code when compared to a known 1D solution. However, in order to be an effective tool for modeling arbitrarily complicated 2D models, we need a way of estimating the solution accuracy at each refinement iteration without a priori knowledge of the solution. We could compute the full solution using all 41 wavenumbers during each refinement, but this would require a very lengthy computational time. We find it is simpler to use the relative differences of $\hat{E}_i$ and $\hat{H}_i$ computed at the receivers between subsequent refinement iterations using only three logarithmically spaced wavenumbers. As shown for the 2D MT problem (Key and Weiss, 2006), this would be unreliable if we only used the basic unweighted error estimator (equation 25), because several refinement iterations can concentrate on regions that have little effect on the solution at the receivers. However, the use of the dual problem and the DEW error estimator (equation 37) focuses the refinement on elements having the largest effect at the receivers. We can therefore expect that the solution accuracy will always be improving at the receivers, and that the relative solution changes between iterations might be of the same order of the solution error. This is supported by the convergence of the relative error for DEW refinement shown Figure 3. Figure 6 shows the mean and maximum relative differences of $\hat{E}_i$ and $\hat{H}_i$ between refinement iterations for the inline geometry. Maximum and mean values were computed using relative differences between subsequent grid refinements for all
receivers and three refinement wavenumbers \((k_x = 0.0001, 0.001, 0.01 \text{ m}^{-1})\). A comparison of Figures 3 and 6 shows that, indeed, the relative differences of \(\hat{E}_x\) and \(\hat{H}_x\) have the same magnitude and character as the actual errors in the full CSEM solution, and are therefore a reasonable proxy for the solution error.

It is also worthwhile to examine the relative differences as a function of receiver position during the refinement iterations, shown in Figure 7 for \(\hat{E}_x\) at \(k_x = 0.0001 \text{ m}^{-1}\). The relative differences exhibit three general phases of character as the grid is refined. During the first phase, refinement results in large changes in the solution at all receiver locations. During phase two, the refinement changes the solution nearest the transmitter at first and then moves out to farther distances with each subsequent refinement. During the last phase, refinement results in small changes over a wide range of positions. The first phase is probably the result of refining the worst elements in the grid, which heavily pollute the solution globally. The second phase is easily explained, because the error estimator is based on gradient recovery. The field gradients and their errors will be largest near the transmitter, and hence, refinement selects this region first.

After refinement, this region contains smaller errors and subsequent refinement moves farther out to positions that still have relatively large gradient errors. Upon entering phase three, the grid has been heavily refined. Further refinement during phase three results in smaller changes in the solution. The three phases can also be identified in the mean and maximum relative difference plots shown in Figure 6. During phase two, the relative differences can actually increase substantially between iterations, which results from refinement moving out to previously unrefined regions farther from the transmitter. However, during phase 3, the solution has a nearly monotonic convergence. Although not shown, \(\hat{H}_x\) and other wavenumbers show a similar behavior.

Figure 3. Convergence of local and DEW-based grid refinement for the canonical 1D reservoir model. The mean relative error of the inline \(H_x\) component at all receivers versus the number of grid vertices is shown for DEW and local refinement using a primary model with either a 1.0 \(\Omega\text{m}\) or 0.3 \(\Omega\text{m}\) seafloor resistivity, as indicated in the legend. Although not shown, a similar plot is observed for the broadside \(E_x\) fields.

Figure 4. Frequency dependency of the finite-element solution error using a grid obtained with DEW refinement for a 0.25 Hz transmission. The mean relative error of the inline \(H_x\) and broadside \(E_x\) at all receivers is shown.

Figure 5. Computation time required to solve the finite-element system for 41 wavenumbers as a function of the number of grid vertices. Computations were performed on an Apple 2.2 GHz G5 computer. Two different linear solvers were used: MFQMR, matrix-free QMR; STQMR, stored matrix QMR (including time to compute and store the matrix \(K\) in equation 24). STLHS is the time to compute and store, the matrix \(K\).

Figure 6. Convergence of the inline \(\hat{E}_x\) and \(\hat{H}_x\) computed at the seafloor EM receivers as the finite-element grid is refined and the number of grid vertices increases. Maximum and mean values were computed using relative differences between subsequent grid refinements for all receivers and three refinement wavenumbers \((k_x = 0.0001, 0.001, 0.01 \text{ m}^{-1})\). See text and Figure 7 for explanation of Phases 1, 2, and 3.
The convergence behavior shown in Figures 6 and 7 suggests the following two-step procedure for estimating when a suitably refined grid has been achieved and the solution has attained the desired accuracy. The first step is to examine the total relative differences of $\hat{E}_i$ and $\hat{H}_i$, as shown in Figure 6. If the desired solution accuracy is 1%, then the maximum relative differences for both $\hat{E}_i$ and $\hat{H}_i$ should be below 1%. However, there is the chance that the relative differences could be in a local minimum of phase two. Therefore, the second step is a visual inspection of the relative differences versus iteration number and receiver positions, to ensure the effects of refinement has moved across the entire range of receivers, as shown in Figure 7.

**EXAMPLE OF A COMPLEX OFFSHORE MODEL**

Here, we present a demonstration of adaptive grid refinement for modeling a complex offshore resistivity structure. Figure 8a shows the model, which includes a resistive salt diapir structure, a 100-m thick resistive oil reservoir located at the apex of an anticlinal sedimentary structure, and a resistive basement of variable depth. For simplification, we only consider a flat seafloor here. In a companion paper (Li and Constable, 2007), the adaptive code is used to study the distorting effects of variable seafloor topography. We consider the case of an inline transmission of 0.1 Hz, located at $y = 6$ km and height 50 m above the seafloor. A linear array of 200 receivers is positioned along the seafloor at $y = -16$ to 20 km. The coarse starting grid consists of 754 vertices and 1461 elements and spans from $z = -10$ to 30 km and $y = -30$ to 50 km. The 1D model for the primary field calculation consists of only air, sea, and a 1-Ωm sediment layer. Adaptive refinement iterations were set to refine 5% of elements with largest DEW error indicator computed at wavenumbers $k_z = 0.00001, 0.0001, and 0.001$ m$^{-1}$. Computations were performed serially on an Apple Xserve G5 dual 2.2 GHz processor machine.

Figure 8b shows the grid after eight refinement iterations. After 31 refinement iterations, the solution converged to a mean relative difference for both $\hat{E}_i$ and $\hat{H}_i$ of about 0.3% and maximum relative differences of about 2%. The final grid consists of 40,154 vertices and 80,233 elements and the central portion is shown in Figure 8c. Near the transmitter, the grid contains the densest concentration of refinement. Closely spaced elements continue out from this region along the seafloor. This is expected since the fields have considerable curvature at the seafloor interface and the DEW error indicator directs refinement toward elements most affecting the dense array of sites along the seafloor. Additional regions of substantial refinement include the oil reservoir, particularly near the left and right sides. Boundaries between the shallowest sedimentary layers show an increased concentration of elements, as well as the corners of the salt diapir that are shallower than about 5-km depth.

The refinement iterations took a total of 120 minutes, while the final grid required 280 minutes to compute the full CSEM solution using all 41 wavenumbers. Of the 6.7 hours total computational time, only about 30% was spent on the refinement iterations. We note that the rather large number of vertices and the long computational time required for this example are biased upwards because we used 200 receivers and ran the grid refinement iterations until the solution was well converged. For the smaller number of receivers that reflects a current CSEM survey (about 40–80), a substantially smaller grid would suffice and correspondingly the computational time would be much shorter. For instance, reducing the number of sites to 100 shorts the computational time to about one hour, with the final refined grid using 24,520 vertices and 48,986 elements.

Figure 8. Evolution of the complex oil reservoir and salt diapir model grid when refined for a linear array of 200 receivers positioned along the seafloor at $y = -16$ to 20 km. (a) Starting model grid (central region only shown) containing 754 vertices and 1461 elements. (b) Fifteenth refined grid containing 4433 vertices and 8818 elements. (c) Final grid computed after 31 refinement iterations, containing 40,154 vertices and 80,233 elements.
Figure 9 shows the convergence of the $\hat{H}_y$ versus receiver position and refinement iteration at $k_e = 0.0001$ m$^{-1}$. Phase one of the refinement is evident in grid numbers 2–6, where the solution contains large changes at most of the receivers. Phase two is evident for grid numbers 7–28, where the refinement yields the largest changes in a band that moves out to increasing distances from the transmitter. During the last two or three iterations, refinement starts moving towards phase three, although there is clearly still some phase two–like behavior at the largest negative positions. The vertical stripes near $y = 2$ and $y = 5$ km result from singularities in $\hat{H}_y$ at this wave-number near these positions. Since the value of $\hat{H}_y$ is minuscule in these regions, the corresponding relative error can sometimes become quite large. At positions less than about $-10$ km, the field amplitudes are below the noise level of most EM transmitter-receiver systems (about $10^{-15}$ V/A.m$^2$) and so the unconverged solution in this region is irrelevant. If only receivers at $y = -10$ to 20 km are considered, the relative differences of $\hat{H}_y$ between the penultimate and final grid have a mean of 0.12% and maximum of 0.33%. If we assume the conclusions from the 1D model study extrapolate to this complex model, the accuracy of the $y$ and $z$ terms for the inline geometry are probably within an order of magnitude of the accuracy of the $x$ term, and so we estimate that $E_x$ and $E_z$ have errors to less than about 3.3%.

Amplitudes of the inline total field components for the central region of the model are shown in Figure 10 along with the primary fields from the background 1D model. The total fields show substantial structure compared to the primary fields, particularly over the reservoir and salt features. The anomalous responses extend well off the resistive features in the direction away from the transmitter. In particular, the $E_y$ component exhibits a decreased amplitude over the resistive features. Off the edges of the resistive features, both $E_y$ and $E_z$ have larger amplitudes than the primary model. The nature of these features is revealed when we consider the polarization of the electric field in a cross section of the model. Figure 11 shows polarization ellipses (e.g., Smith and Ward, 1974) for both the total and primary electric field in the central region of the model. Shaded colors correspond to the magnitude of the total field polarization ellipse major axis. The magnitude shows amplified electric field strength in both the reservoir and salt diapir. The ellipse orientations show that the resistive features impede the flow of electric currents. Where the primary field ellipses show vertical current flow in the region above both resistors, the total fields are bent strongly to the horizontal plane, resulting in the reduced $E_z$ observed at the receivers. Along the lateral edges of the resistors, the fields curve rapidly around as vertical current flow is no longer impeded. For instance, at a position of 0 km, the total field ellipses warp around the left edge of the reservoir and a strong vertical field is also observed at the seafloor receivers. Since the resistors impede the flow of vertical current, there is always a net increase of current flowing around the resistors, which eventually turns downward after traveling beyond the end of the resistor. This explains why $E_z$ in Figure 10 has an increased amplitude well beyond the far ends of both resistors. A small decrease in $E_y$ is observed at ranges of about 9–11 km. This correlates with the total field ellipses bending back around the upper left section of the salt diapir. It appears that the currents are doubling back around the diapir before diving down along its left edge. The reversed direction of the currents around this region of the salt results in a lower magnitude $E_y$ on the seafloor. Inside the reservoir, the electric field is linearly polarized in the vertical direction as a result of...
of surface, or galvanic, charges on the upper and lower surfaces of the reservoir. Surface charges are predicted from the conservation of the normal component of current across an interface (e.g., Stratton, 1941, p. 483). The low magnitude of $E_z$ above the reservoir is due in part to a strong secondary field (not shown) associated with surface charges on the reservoir. The secondary field is in reverse polarity to the primary field and results in a reduced total field amplitude above the reservoir. The effects of surface charges are also evident on the inside edges of the salt diapir where the electric polarization is normal to the edge boundaries. Other features of the model, including different transmitter locations and frequencies, could be investigated and we anticipate future model studies using the adaptive finite-element algorithm.

CONCLUSIONS

The adaptive finite-element method using unstructured grids is a new tool for modeling complex 2D structures for the frequency-domain marine controlled-source electromagnetic method. The use of an unstructured mesh allows for arbitrarily complex 2D structures to be included in numerical models. The matrix-free QMR method can be used to solve the linear FE system of equations and requires much less storage than full matrix inversion methods and allows for computation of large finite-element grids. However, the matrix-free QMR requires about 20% longer computational time than a sparsely stored matrix version of QMR. The adaptive grid refinement scheme uses a dual-weighted a posteriori error estimator and allows for grid refinement in a robust and efficient manner. Adaptive refinement presents a method for automatically computing asymptotically exact solutions to the 2.5D CSEM partial differential equations since refinement can be continued until the solution converges to the desired level of accuracy. We have found that refining about 5% of elements per iteration gives a good rate of convergence without over-refining the grid or greatly increasing the computational time per iteration.

The validation tests show that simple relative differences of $\hat{E}_x$ and $\hat{H}_x$ are a reasonable proxy for the magnitude of the solution error. The relative differences exhibit three phases of character during the refinement iterations. Since the mean and maximum relative differences can exhibit local minima during phase two of the refinement iterations, it is worthwhile to inspect the visual characteristics of the relative differences to ensure that refinement has progressed past phase two.

The offshore model study illustrates the kinds of structures that can be accurately modeled with this code. The companion paper (Li and Constable, 2007) presents a systematic study of the effects of seafloor bathymetry variations on CSEM measurements. The adaptive refinement presented here could easily be implemented into inversion algorithms and with some effort could be extended to 3D EM methods. We anticipate this code will prove useful for future offshore model studies of both simple and complex offshore structures.

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REFERENCES


Ramacher, R., and F. Stuttmeier, 1998, A posteriori error control in finite el-
element methods via duality techniques: Application to perfect plasticity: Computational Mechanics, 21, 123–133.
Unsworth, M. J., B. J. Travis, and A. D. Chave, 1993, Electromagnetic induction by a finite electric dipole source over a 2-D earth: Geophysics, 58, 198–214.